113 Class Problems: Finite Symmetric Groups

1. Let $\sigma = (1264)(39), \tau = (12)(57)(89) \in Sym_n$. Write $\sigma\tau$ in disjoint cycle notation.

   Solution:

$$\sigma\tau = (1264)(39)(12)(57)(89) = (164)(2)(398)(57)$$

2. Consider the cycle $(123456) \in Sym_n$.

   (a) Write $(123456)^d$ in disjoint cycle notation for all $d \in \{1, 2, 3, 4, 5, 6\}$.

   (b) What do you notice about the cycle structures in part (a)? If $\sigma \in Sym_n$ is a cycle of length m can you conjecture the possible cycle structures of σ^d where $d \in \{1, 2, \cdots, m\}$? Can you prove your conjecture?

   Solution:

Cycle structure

a) $(123456)^1 = (123456) \longrightarrow \{6\}$

$(123456)^2 = (135)(246) \longrightarrow \{3,3\}$

$(123456)^3 = (14)(25)(36) \longrightarrow \{2,2,2\}$

$(123456)^4 = (153)(264) \longrightarrow \{3,3\}$

$(123456)^5 = (165432) \longrightarrow \{6\}$

$(123456)^6 = (1)(2)(3)(4)(5)(6) \longrightarrow \{1,1,1,1,1,1\}$

b) They are all partitions of 6 with equal terms

length of cycle

Conjecture : $(1 \ldots m)^d$ has cycle structure a partition of m with equal terms

<u>Proof</u> If $(1 \ldots m)^d$ has cycle structure $\{k_1, k_2, \ldots, k_r\}$

then 1/ $k_i = k_j \ \forall i,j$ as $(123\ldots m) = (23\ldots m1) = \ldots = (m12\ldots m-1)$

2/ $k_1 + \cdots + k_r = m$

3. (Hard) Let S be everyone in the class, including the Professor Paulin. Imagine we have a machine that can swap any two peoples' minds. Two people sit in it and boom, their minds are swapped. The catch is that, as a pair, they cannot use the machine again. Now imagine we go crazy and use it many times. Because transpositions generate Sym_n, our minds could end up in any permutation.

   Is it possible to somehow put everyone's mind back in the correct body? Definitely not in general. Maybe every single possible pair has used the machine.

   Prove the following: If we introduce two extra people $\{x, y\}$, then it is always possible to put everyone's mind back in their own body.

   For example, if we've only used the machine once, giving the transposition (12), then

$$(xy)(y1)(x2)(y2)(x1)(12) = Id_{S\cup\{x,y\}}.$$

   Hint: Look at the episode of Futurama called *The Prisoner of Benda.* That's where this ingenious problem is from. You'll even find a proof.

Assume $|S| = n \in \mathbb{N}$. Let $\sigma \in Sym_n$. Then we can express σ as a product of disjoint cycles. Hence it is enough to solve the problem for a single cycle.

Observe

$$(y1)(x2)(yk)\dots(y3)(y2)(x1) = (123\dots k)(xy)$$

$$\Rightarrow \underbrace{(x1)(y2)(y3)\dots(yk)(x2)(y1)}_{\text{Should do this to undo } (123\dots k)}(1234\dots k) = (xy)$$

x and y are flipped

Strategy: Put σ in disjoint cycle notation. Apply these transpositions to each cycle. If at the end x and y are flipped, apply (xy).